The Poincaré-Hopf theorem

MM7020 Mathematical communication — Assignment 2

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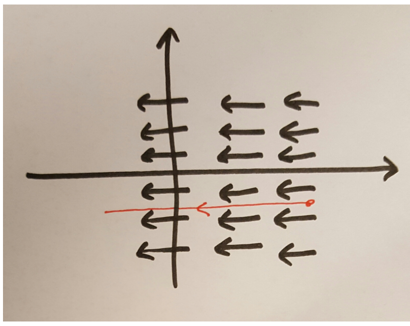
Assuming standard knowledge of mathematics up to multivariable calculus we give an intuitive exposition of vector fields on manifolds, the main exhibit being the hairy ball theorem for the sphere S^2 and its generalisation to compact manifolds due to Poincaré and Hopf.

1 Vector fields and the hairy ball theorem

A vector field on $\mathbf{R}^n$ is usually seen as a function $f : \mathbf{R}^n \to \mathbf{R}^n$ with some assumptions of continuity. For our intents and purposes we will assume all vector fields infinitely differentiable, also known as *smooth*. Vector fields are widely used in physics in order to mathematically describe the velocity in a fluid, electromagnetic fields, gravity, etc.



(a) Field A: constant field

(b) Field B: spirals approaching the unit circle

Figure 1: Two vector fields on $\mathbf{R}^2$ with integral curves in red: this is the curve describing the motion of a "point particle" under the influence of the vector field over time.

We say that a vector field f *vanishes* if for some point $p \in \mathbf{R}^n$, we have $f(p) = 0$. Clearly field B above vanishes at $(0,0)$ (the origin is a stationary point) while field A does not vanish on $\mathbf{R}^2$ (it is constant).

From here it is not very difficult to imagine vector fields on more general shaped domains. A nice class of examples is that of *smooth level surfaces*: for some real value a and some smooth real-valued function $h : \mathbf{R}^3 \to \mathbf{R}$, consider $S = h^{-1}(\{a\})$, the set of points it maps to a. Under the assumption that ∇h is nonzero throughout S, this is a smooth surface.

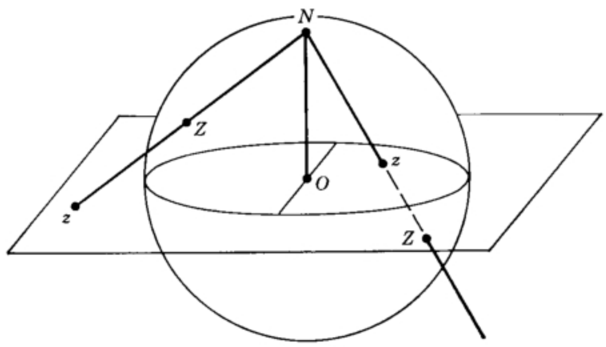
At every point of S we may consider the plane of vectors of $\mathbf{R}^3$ which are tangent to the surface - this is the 2-dimensional *tangent space* of S at the point. Consequently a smooth vector field on S is the assignment of a tangent vector at every point of S in a way that varies smoothly along the surface.

Two crucial examples of surfaces that we will consider here are the *sphere* S^2 - the set of unit vectors in 3-space[1] and the *torus* T^2. The latter is maybe not very familiar in equation form[2] but we have a very good intuition for it being a doughnut-shaped hollow surface.

Let us focus on the sphere for a bit. The following mysterious sounding statement is incredibly useful in studying S^2:

the sphere with one point removed looks like the real plane.

In fact, let N be the north pole of our sphere, i.e. $N = (0, 0, 1)$. Now select a point $Z \in S^2 \setminus \{N\}$ and draw the line between Z and N. This line intersects the "real plane" $\mathbf{R}^2 \times \{0\} = \{(x, y, 0)\}$ at exactly one point z.



Writing out this assignment $Z \mapsto z$ we see that it is in fact invertible and smooth as we vary Z, as long as we avoid N. (The assignment is a smooth bijection between $S^2 \setminus \{N\}$ and $\mathbf{R}^2$, usually called the *stereographic projection*.) We also see that avoiding N was truly necessary - as we travel to the north pole along the sphere the corresponding points z of $\mathbf{R}^2$ will be removed very far away so that the sphere's north pole morally corresponds to a "point at infinity" for $\mathbf{R}^2$.

In this sense, a lot of what happens in $\mathbf{R}^2$ can be extended to the sphere by just "adding behaviour at ∞". Now let's extend the vector fields in Figure 1 to the sphere.

In the original field B in Figure 1b, if a particle started out in the origin $(0, 0)$ it stayed there "forever" (this was a zero for the vector field). Particles placed in $\mathbf{R}^2$ outside and inside the unit circle travelled towards the unit circle. This "fixes" the

[1]This is a smooth level surface for $(x, y, z) \mapsto \|(x, y, z)\|$ and $a = 1$

[2]A torus with radii $0 < r < R$ is a level surface for $(x, y, z) \mapsto (R - \sqrt{x^2 + y^2})^2 + z^2$ and $a = r^2$

point at infinity: a particle starting out infinitely far away remains so. We thus have a new zero in the corresponding vector field on S^2 - the north pole:

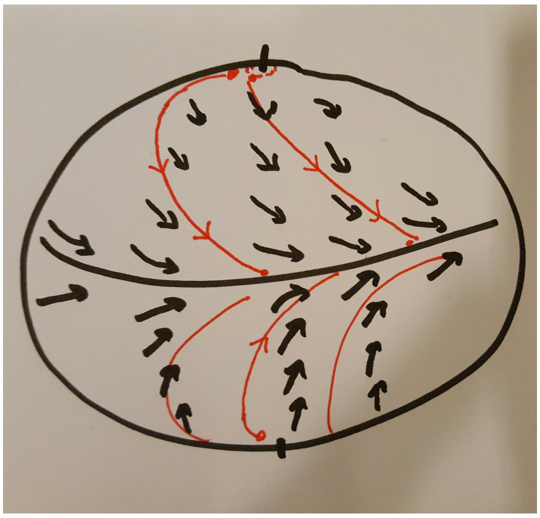
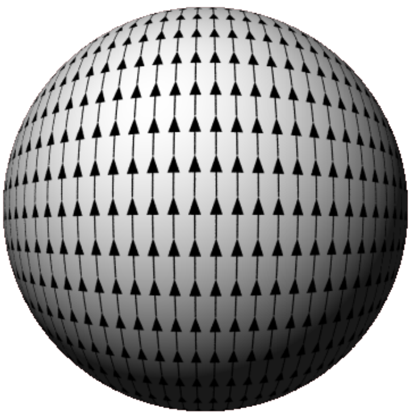
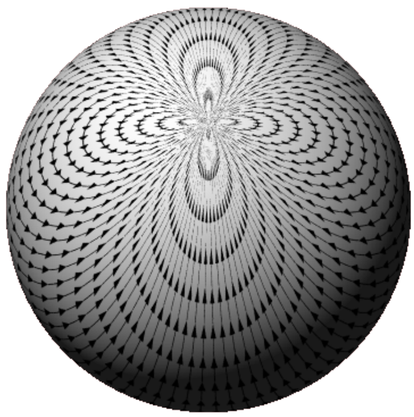


Figure 2: Field B extended to the sphere. We see that the unit circle of $\mathbf{R}^2$ corresponds to the equator of the sphere. What was outside the unit circle corresponds to the upper hemisphere and what was inside - to the lower hemisphere. The origin $(0,0)$ corresponds to the south pole $(0,0,-1)$.

The vector field A in Figure 1a translated every point of $\mathbf{R}^2$. However, if we fold up the plane using the stereographic projection above we see that as we get further away from the origin (and closer towards the north pole N) this constant amount of translation, as seen on the sphere, matters less and less. Taking this argument to the extreme, this fixes the point at infinity. That is, even though we tried, we could not create a vector field on the sphere without zeroes.



(a) The $\mathbf{R}^2$ part



(b) Behaviour at infinity

Figure 3: Field A extended to the sphere

Judging by the examples above, and also intuitively, it seems difficult to "comb a hairy ball without creating any licks". It turns out that this is not only difficult, but mathematically impossible. This is the content of the famous *hairy ball theorem*:

Theorem 1.1. Any vector field on S^2 vanishes.

Proof. This can be deduced in an elementary way from a more general result due to Poincaré and Hopf, to which we dedicate Section 3. □

Now consider the torus T^2. We can easily comb it without creating licks - for instance, like this:

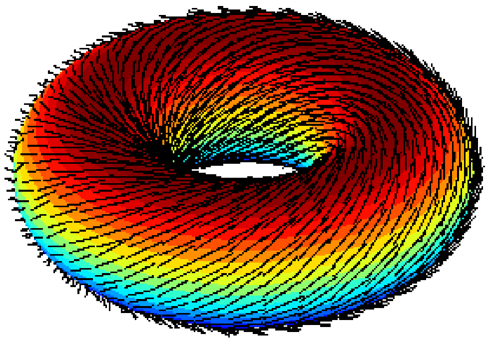


Figure 4: Combed torus

How come one can easily find nonvanishing vector fields on the torus but not on the sphere? After all the two are similar as objects - they are both two-dimensional surfaces and they are both compact[3]! Well, the obvious difference is that the torus has a hole where the sphere doesn't: *we cannot stretch and bend the sphere into the shape of a torus*. It turns out that it is this sort of geometrical property that forces any vector field on the sphere to have zeroes. Let us briefly define some concepts in order to quantify this difference as we move on towards an appropriate generalisation of Theorem 1.1.

2 Some technology needed

The proper setting of object in which to generalise 1.1 is really not much different from the smooth level surfaces discussed earlier. A *smooth k-manifold*[4] is a subset $M \subseteq \mathbf{R}^n$ of Euclidean n-space that *locally* (i.e. if you look close enough) looks like a piece of a smooth k-dimensional level surface (i.e. the preimage of an a under

[3]Closed and bounded.

[4]This is the historical definition of (smooth) manifold (*variété* in French) given by Henri Poincaré at the turn of the 20th century. The modern definition usually involves more abstract machinery but is equivalent to the one given here.

a nice function[5]). The tangent space of M at a point is defined in the exact same way as before and is again k-dimensional. The definition of manifold allows for capturing many properties of "smooth" objects in a *coordinate free* way.

Of course, letting $n = 3$ and $k = 2$ in this definition includes the case of smooth level surfaces of Section 1.

Now let us briefly return to the $(n, k) = (3, 2)$ setting of the previous section and the geometry of surfaces. NB: in this context, "surface" refers to a 2-dimensional manifold, but a "level surface" can be of any dimension!

The concept of a *triangulation* is best understood as a combinatorial approximation of a smooth object. (A reader familiar with computer modelling can think of it as meshing!) Slightly more formally, it is a homeomorphism between the manifold and a simplicial complex, or intuitively a continuous deformation of the manifold into a polytope, as illustrated in Figure 5. If a manifold has a triangulation we say it is *triangulable*.

In a triangulation, we call an n-dimensional "piece" of the polytope in question an *n-simplex* (in plural *simplices*). That is, a 0-simplex is a point, a 1-simplex is a line, a 2-simplex a face, etc.

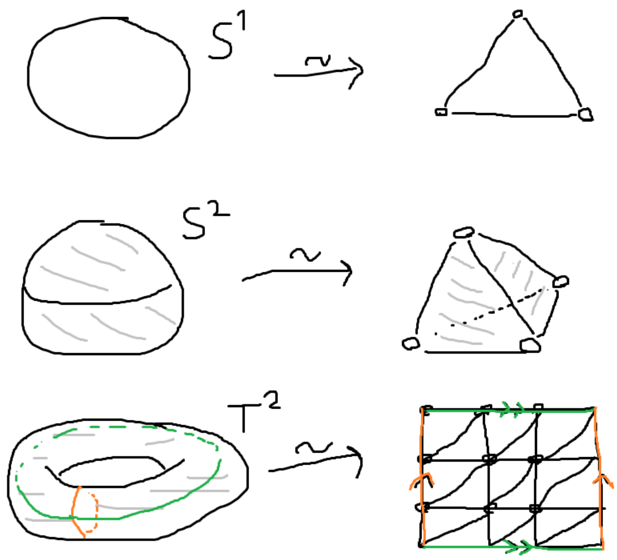


Figure 5: examples of triangulations

Definition 2.1. The *Euler characteristic* $\chi(M)$ of a triangulable manifold is the alternating sum of the number of n-simplices in its triangulation.

[5]Now the function h of the level surface is defined on (an open set of) $\mathbf{R}^n$ and takes values in $\mathbf{R}^{n-k}$.

It can be shown that the characteristic does not depend on the choice of triangulation. The interested reader may also find that it can be defined through the dimensions of the so-called homology groups, the *Betti numbers*. For the familiar case of closed orientable surfaces, two examples of which we have already seen, this has a nice expression:

Proposition 2.2. $\chi(S) = 2 - 2g$ for a closed orientable surface S, where g is its *genus* (in plural *genera*), i.e. the number of holes. In particular $\chi(S^2) = 2$ and $\chi(T^2) = 0$.

Proof. See for instance [Rey89]. □

A classical result describes the surface of a polyhedron with V corners, E edges and F faces having the Euler characteristic $\chi(S) = V - E + F$. For a convex polyhedron, $\chi(S) = 2$. This confirms our intuition - any convex polyhedron can be deformed into a sphere, which has $\chi = 2$ by Proposition 2.2.

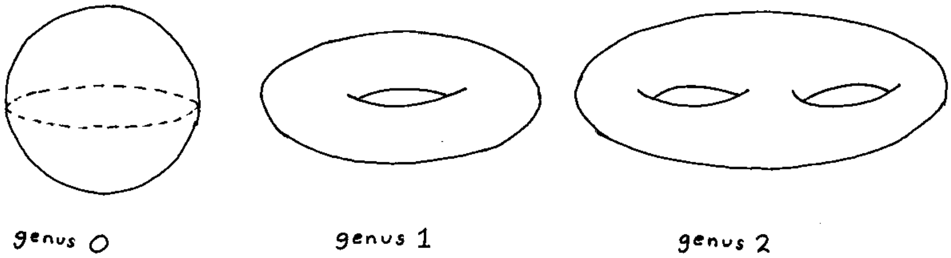


Figure 6: some orientable closed surfaces and their genera

Definition 2.3. The *n-sphere* S^n is the set of unit vectors in $\mathbf{R}^{n+1}$. It is naturally a level surface of the polynomial function $\| \cdot \|$ and as such an n-dimensional (closed) manifold.

The natural triangulation of the n-sphere is of course the n-simplex. By combinatorial reasoning we can find that its Euler characteristic is given by the alternating sum of terms $\binom{n}{k}$. From this, the following result will eventuate.

Proposition 2.4. $\chi(S^n) = 1 + (-1)^n$.

Similarly to how a root of a polynomial has a well-defined multiplicity, we can define an *index* of a zero for a vector field. If the zero is isolated we can pick a ball around it without any other zeros within it. By restriction to the edge of this ball (and normalisation), the vector field will define a map to the unit sphere in $\mathbf{R}^n$ and for this map we can count the (integer) number of preimages. This intuitively corresponds to the "swirliness" of the vector field, or "how much" it whirls around the zero. For a point p and a vector field v we denote this (integer) number by $\mathrm{ind}_p(v)$.

3 Poincaré-Hopf

With this definition of *index* above we are ready to state the theorem in full:

Theorem 3.1. For a compact manifold M, the equation

$$\sum_{p \in M} \operatorname{ind}_p(v) = \chi(M)$$

holds for any smooth vector field v.

Proof sketch. Viewing M as a subset of some Euclidean space $\mathbf{R}^n$, we can construct a "thickening" N and use the so-called Gauss map from N to the unit sphere. The mapping degree of this map can be shown to be the sum of the indices and most importantly *independent of the choice of vector field*. Then it is possible to construct an explicit vector field from a given triangulation for which the sum of indices explicitly equals the characteristic.

For a complete proof, see a book, for instance [AH74] or [Mil97]. □

Vector fields are analytical objects developed mainly from physics, and the Euler characteristic is a geometrical property constructed using combinatorics. The fact that these two things, each from a different branch of mathematics, turn out to be related in such a fundamental way is perhaps surprising at first, but beautifully demonstrates that there are deep properties underlying these concepts. We will not discuss this further philosophically, however it is a strong point of argumentation that these branches of mathematics are actually describing something meaningful and important about our world.

Poincaré's original theorem and the generalisation by Hopf, Theorem 3.1, are some of the first of many examples of results connecting these areas. These two branches once seemed far apart, but now they can be viewed as different aspects of a common general framework. Some immediate consequences include the Hairy ball theorem, Theorem 1.1, as promised in Section 1 and settling the question regarding vector fields on closed surfaces.

Corollary 3.2. Any vector field on a sphere S^n for $n \in 2\mathbf{Z}$ vanishes.

Proof. $\chi(S^{2n}) = 2$ by Proposition 2.4, and in particular the sum

$$\sum_{p \in M} \operatorname{ind}_p(v) > 0$$

in Theorem 3.1 cannot be an empty sum. Hence the vector field v must have at least one zero. □

Corollary 3.3. If a closed orientable surface S has a nonvanishing smooth vector field, then it is a torus.

Proof. By Proposition 2.2, the Euler characteristic of S is given by $\chi(S) = 2 - 2g$. If S has a nonvanishing vector field, then by Poincaré-Hopf $\chi(S) = 2 - 2g = 0$ so $g = 1$ (the surface has one hole). □

References

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