The étale fundamental group, étale homotopy and anabelian geometry

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Lecture notes

This is a typed-out and slightly expanded version of my notes that I made whilst preparing the presentation of my thesis [Sar17] which took place on Aug 23, 2017 at KTH. The two main sources of inspiration for the exposition are Szamuely's book [Sza09] which covers Galois theory, covering theory and the étale fundamental group - although without Galois categories - and a lecture at the conference "Motives, algebraic geometry and topology under the white-blue sky" in Munich on July 6, 2017 where Alexander Schmidt presented the paper "Anabelian geometry with étale homotopy types", [SS16].

The presentation is going to proceed in the following manner: first we are going to give two classical examples to illustrate the concept of a Galois category, which we are defining in section 3. In the section following that, we explain how this formalism gives the étale fundamental group of a scheme. After that, we will discuss a series of advanced results that uses this group, and describe some conjectures that are central for researchers in this area. Then, after a small but necessary technical interlude, we will present some recent results of a slightly more general nature.

1. Galois theory

Let k be a field. A finite dimensional k-algebra A is *étale* if it is isomorphic to a finite product of separable extensions K_i of k,

$$A \cong \prod_{i=1}^{n} K_i.$$

Given a separable closure k_s of k, the absolute Galois group $\mathrm{Gal}(k_s|k)$ acts on the finite set $\hom_k(A, k_s)$. Sending finite étale algebras A to finite sets with a $\mathrm{Gal}(k)$-action $\hom(A, k_s)$ is a contravariant functor

$$\begin{aligned} (\text{finite étale } k\text{-algebras})^{\mathrm{op}} &\xrightarrow{\ F\ } (\text{finite left } \mathrm{Gal}(k)\text{-sets}) \\ A &\longmapsto \hom_k(A, k_s) \end{aligned}$$

and we have a theorem:

Theorem 1.1 (Main theorem of Galois theory)**.** *For k a field, F defines an anti-equivalence of categories,*

$$(\text{finite étale } k\text{-algebras}) \simeq (\text{finite continuous left } \mathrm{Gal}(k)\text{-sets}).$$

Remark 1.2. Some things that we can note:

- The absolute Galois group $\mathrm{Gal}(k_s|k)$ implies a choice of separable closure k_s.
- k_s is not a finite étale algebra, but it is a limit of finite étale algebras. In fact, it is the union of all finite separable extensions of k. Similarly $\mathrm{Gal}(k_s|k)$ is not a finite group, but it is a limit of the finite groups $\mathrm{Gal}(L|k)$ for all intermediate finite Galois extensions L of k in k_s.

- $\operatorname{Aut} F \cong \operatorname{Gal}(k_s|k)$. We will explain why this is interesting.

We also have some general theorems showing that we can classify certain classes of fields by their absolute Galois groups. These are "anabelian" results predating the word anabelian!

Theorem 1.3 (Neukirch 1969)**.** *Let K, L be algebraic number fields. If $\operatorname{Gal}(K) \cong \operatorname{Gal}(L)$ then $K \cong L$.*

Theorem 1.4 (Uchida 1973)**.** *For K an algebraic number field, the outer isomorphisms of the Galois group corresponds to the automorphisms of the field:* $\operatorname{Aut}(K) \cong \operatorname{Out}(\operatorname{Gal}(K))$.

2. Covering theory

Let $X \in \mathbf{Top}$ be a "nice" (i.e. connected, locally connected and locally simply connected) space.

A *covering* of the space X is a pair $f : Y \to X$ of a space Y and a continuous function f such that f is a local homeomorphism, admitting a cover of open sets such that the preimage of such an open set U consists of disjoint open sets mapped homeomorphically to U. Thus the fibre of a covering map is a discrete set.

Given a cover Y of X, the fundamental group $\pi_1(X, x)$ acts on the fibre $p^{-1}(x)$ by the *monodromy action*, which is defined in the following way: we begin by choosing an element $\alpha \in \pi_1(X, x)$ which is represented by a loop $\gamma : I \to X$. By choosing a start point y above x, we get a lifting $\tilde{\gamma} : I \to Y$. In general this will not be a loop but a path from y to another point in the fibre over x, and this defines a permutation of the fibre, hence a group action. Taking a covering $p : Y \to X$ to the set $p^{-1}(x)$ is a functor Fib_x

$$\begin{aligned} (\text{covers of } X) &\xrightarrow{\operatorname{Fib}_x} (\text{continuous left } \pi_1(X, x)\text{-sets}) \\ (p : Y \to X) &\longmapsto p^{-1}(x) \end{aligned}$$

The following theorem is remarkable because it resembles Galois theory:

Theorem 2.1 (Main theorem of covering theory)**.** *For a connected, locally connected and locally simply connected topological space X with base point x, the fibre functor Fib_x sending a covering p to the fibre $p^{-1}(x)$ is an equivalence of categories*

$$(\text{coverings of } X) \simeq (\text{left } \pi_1(X, x)\text{-sets}).$$

Here connected coverings give sets with transitive action and Galois coverings give coset spaces of normal subgroups. If we restrict ourselves to *finite covers*, i.e. where the fibres are finite sets, we get another equivalence of categories

$$(\text{finite coverings of } X) \simeq (\text{continuous left } \widehat{\pi_1(X, x)}\text{-sets})$$

where the latter notation denotes the profinite completion of $\pi_1(X, x)$, which is obtained as a limit over the system of fundamental group all finite covers — just like the absolute Galois group!

Remark 2.2. A nice space X with base point x has a universal covering space, whose defining property is that the cover $\tilde{X}_x \to X$ factors through all other covers $Y \to X$ and thus that $\pi_1(X, x) \cong \operatorname{Aut}(\tilde{X}_x)$. One can verify that the fibre functor Fib_x is represented by the universal covering space, so that $\operatorname{Fib}_x(Y) = \hom_X(\tilde{X}_x, Y)$. In general, the universal covering is not a finite cover, but it can be described as a limit of all finite connected covers.

3. Grothendieck's Galois theory

We have now seen two examples of classical subjects with a central theorem stating the equivalence of certain interesting objects (algebras, coverings) correspond to sets with a group action. Grothendieck developed a beautiful common generalisation in SGA 1, V.5.1, where he defines a *Galois category*.

Definition 3.1. A *Galois category* is a category $\mathcal{C}$ which is equivalent to the category of finite continuous left G-sets for some profinite group G,

$$\mathcal{C} \simeq G\text{-sets}.$$

where G is referred to as *the fundamental group of* $\mathcal{C}$.

An equivalent formulation is that we have a functor $F : \mathcal{C} \to \mathbf{Set}$ called the *fibre functor* such that $G = \operatorname{Aut} F$ and which lifts to an equivalence $\mathcal{C} \simeq G$-sets as

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathbf{Set} \\ & \searrow^{\simeq} & \uparrow U \\ & & G\text{-sets.} \end{array}$$

Proposition 3.2 (Stacks project 0BMQ)**.** *$(\mathcal{C}, F)$ being a Galois category is equivalent to the following conditions.*

1. *$\mathcal{C}$ is finitely cocomplete and complete.*
2. *Every $X \in \mathcal{C}$ is a coproduct of connected objects.*
3. *Every FX is a finite set.*
4. *F is exact and conservative.*

Here finitely complete and cocomplete means having all finite limits and colimits, and F being exact means that it preserves all finite limits and colimits. A connected object Y is one for which having a monomorphism $Z \to Y$ implies that Z is initial or that $Z \cong Y$.

4. For schemes

We have now arrived at what could be seen as the main topic of this talk, which is to discuss a very important example of a Galois category.

Let X be a connected scheme, and let $\mathbf{FEt}_X$ be the category of all schemes Y with a fixed finite étale map $p : Y \to X$. If we let $\overline{x} : \Omega \to X$ be a geometric point we have the fibre $p^{-1}(\overline{x})$ as the underlying finite set of the pullback

$$\begin{array}{ccc} p^{-1}(\overline{x}) & \longrightarrow & Y \\ \downarrow & & \downarrow p \\ \Omega & \xrightarrow{\overline{x}} & X. \end{array}$$

This defines a functor $\mathrm{Fib}_{\overline{x}}$ taking a finite étale cover Y to the finite set $p^{-1}(\overline{x})$. One can show that it satisfies conditions 1-4 of 3.2 and thus qualifies as a fibre functor making $\mathbf{FEt}_X$ a Galois category:

$$\begin{array}{ccc} \mathbf{FEt}_X & \xrightarrow{\mathrm{Fib}_{\overline{x}}} & \text{(continuous left } G\text{-sets)} \\ (p : Y \to X) & \longmapsto & p^{-1}(\overline{x}) \end{array}$$

Definition 4.1. The *étale fundamental group* is the group $\pi_1^{\text{ét}}(X, \overline{x}) = G = \mathrm{Aut}(\mathrm{Fib}_{\overline{x}})$.

Examples 4.2. Two interesting examples:

- Finite étale covers of the spectrum of a field correspond to finite étale algebras. With $X = \operatorname{Spec} k$ and $\overline{x} : \operatorname{Spec} \overline{k} \to X$ we have $\pi_1^{\text{ét}}(X, \overline{x}) = \mathrm{Gal}(\overline{k}|k)$. Here the choice of closure $\overline{k}$ is a choice of base point. We recover classical Galois theory in a geometric way!
- X finite type over $\mathbb{C}$: we have an anlytical topology and

  $$\widehat{\pi_1^{\text{top}}(X^{\text{an}}, \overline{x})} \cong \pi_1^{\text{ét}}(X, \overline{x}).$$

  This means that we already know many étale fundamental groups. For instance

  $$\pi_1^{\text{ét}}(\mathbb{A}^1_{\mathbb{C}} - 0) = \widehat{\mathbb{Z}}.$$

Remark 4.3. Let us compare with the remarks from the first two sections. In general, there is no possibility to define a universal finite étale cover for a scheme. However, we can construct a system of finite étale covers $(X_\alpha \to X)$ which is a *pro-representing* system for the étale fundamental group, meaning that

$$\varinjlim_{\alpha} \hom(X_\alpha, Y) \cong \mathrm{Fib}_{\overline{x}}(Y).$$

Since every X_α is a finite étale cover with a finite automorphism group, this implies that the automorphism group of the functor is obtained as a limit of this system of finite groups. This is the (Galois category-free) approach used by [Sza09] to show that the étale fundamental group is profinite.

5. Short homotopy exact sequence

Just as in topology, maps between schemes and changes of base point induce homomorphisms of fundamental groups. One of the most important examples is the following. For X a scheme over a field k, we have natural maps $X_{\overline{k}} \to X \to \operatorname{Spec} k$.

Theorem 5.1 (Short homotopy exact sequence)**.** *For X geometrically connected, quasiseparated and quasicompact, we have a short exact sequence*

$$1 \longrightarrow \pi_1(X_{\overline{k}}, \overline{x}) \longrightarrow \pi_1(X, \overline{x}) \longrightarrow \mathrm{Gal}(k) \longrightarrow 1.$$

The group $\pi_1(X_{\overline{k}})$ is called the *geometric fundamental group* of X. Constructing the inner and outer automorphism groups we get

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X_{\overline{k}}, \overline{x}) & \longrightarrow & \pi_1(X, \overline{x}) & \longrightarrow & \mathrm{Gal}(\overline{k}|k) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}(\pi_1(X_{\overline{k}}, \overline{x})) & \longrightarrow & \mathrm{Aut}(\pi_1(X_{\overline{k}}, \overline{x})) & \longrightarrow & \mathrm{Out}(\pi_1(X_{\overline{k}}, \overline{x})) & \longrightarrow & 1 \end{array}$$

which defines a natural continuous map

$$\rho_X : \mathrm{Gal}(\bar{k}|k) \to \mathrm{Out}(\pi_1(X_{\bar{k}})) = \mathrm{Aut}(\pi_1(X_{\bar{k}}))/\mathrm{Inn}((\pi_1(X_{\bar{k}}))$$

called the *outer Galois representation*, which is very important in number theory. For instance, if $k = \mathbb{Q}$, it gives information about $\mathrm{Gal}(\overline{\mathbb{Q}}|\,\mathbb{Q})$ which is very desirable for a number theoretician.

6. Anabelian geometry

In 1983 Grothendieck wrote a letter to Faltings. The letter was concerned with so-called *anabelian* (German: *anabelsch*) schemes over a field k that is a finitely generated extension of $\mathbb{Q}$. Although he does not give a definition of what an anabelian scheme X is, Grothendieck claims that they exist and that that they should be reconstructible from their étale fundamental group $\pi_1(X)$. He makes three conjectures.

Conjectures 6.1. *The three conjectures are as follows:*

1. *If C is a smooth hyperbolic curve (i.e. one with negative Euler characteristic) then C is anabelian.*
2. *If X is a smooth scheme, then every point of X has a fundamental system of Zariski neighbourhoods that are anabelian.*
3. *If X is anabelian the rational points $X(k)$ (= sections of the structure map $X \to \mathrm{Spec}\, k$) are reconstructible from the Galois sections of $\pi_1(X) \to G_k$.*

Here G_k is the absolute Galois group of k. The third conjecture is sometimes referred to as *the section conjecture*. We will otherwise refer to them as conjecture 1, 2 and 3 and focus mainly on conjecture 1. One fundamental issue in interpreting these conjectures, is to say what the word *reconstructible* means. Here are three possible meanings for X, Y smooth:

- ("weak") If $\pi_1(X) \cong \pi_1(Y)$ then $X \cong_k Y$.
- ("strong") The map $\mathrm{Isom}_k(Y, X) \to \mathrm{Isom}_{G_k}(\pi_1(Y), \pi_1(X))/\pi_1(X_{\bar{k}})$ is an isomorphism.
- ("Yoneda") For every Y the map $\mathrm{Hom}_k^{\mathrm{dom}}(Y, X) \to \mathrm{Hom}_{G_k}^{\mathrm{open}}(\pi_1(Y), \pi_1(X))/\pi_1(X_{\bar{k}})$ is an isomorphism.

Tamagawa showed conjecture 1 with the "strong" condition for affine curves, and Mochizuki subsequently improved on that result. Mochizuki also showed conjecture 1 for a larger class of curves for the "Yoneda" condition. See [Tam97] and [Moc99].

Here we will discuss a proof of conjecture 1 with the "weak" condition generalised for a large class of higher dimensional schemes. To do this, we need to make a small detour to introduce some machinery. We will not discuss conjecture 2, but we can note that a version thereof is also proven in [SS16].

7. Technical interlude

Observation: it is not very plausible to have reconstruction from the fundamental group π_1 unless the scheme is of type $K(\pi, 1)$, so it makes more sense to ask a modified question. It is natural to instead look at the étale homotopy type $X_{\mathrm{et}} \in \mathrm{Ho}(\text{pro-}\mathbf{ss})$.

Recall that for any category $\mathcal{C}$ the pro-category pro-$\mathcal{C}$ is defined with

- Objects: functors $X : I^{\mathrm{op}} \to \mathcal{C}$ where I is small and filtering.

- Morphisms: $\mathrm{Hom}_{\text{pro-}\mathcal{C}}((X_i),(Y_j)) = \lim_i \operatorname{colim}_j \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j)$.

Example 7.1. The category of profinite groups can be viewed as the pro-category of finite groups.

Example 7.2. For the category of simplicial sets **ss** (= spaces) has a pro-category pro-**ss** and there is a pointed category pro-$\mathbf{ss}_*$. For a pro-space $X = (X_i) \in$ pro-**ss** we can use the homotopy group functors π_n level-wise and obtain a pro-group $\pi_n(X) = (\pi_n(X_i))$.

The category of pro-spaces has a model structure, defined by Isaksen in [Isa01]. It is defined by

- Cofibrations: (isomorphic to) level-wise cofibrations.
- Weak equivalences: "isomorphisms on homotopy groups".

This latter statement is formalised with the notion of local systems, and the class of fibrations is obtained by the model category requirements from the other two classes.

Remark 7.3. (from [Sch17]) "Pro-spaces are beasts":

- They do not always have points. A point is a morphism $* \to X$. The space $X_n = ([n, \infty))$ has no points, as any image of a constant function will be outside the interval $[n, \infty)$ for large enough n.
- The homotopy groups $\pi_n(X, x)$ may depend on the basepoint $x \in X$ even if X is connected. Thus questions of the type "unpointed versus pointed" are subtle in $\mathrm{Ho}(\text{pro-}\mathbf{ss})$! These problems are treated in section A.2.2 of [SS16].

7.a. Étale homotopy

The étale homotopy type was defined by Artin-Mazur in [AM86] and improved by Friedlander in [Fri82]. For a locally Noetherian scheme X, the *étale homotopy type* is an object X_{et} in pro-**ss** defined as taking all rigid hypercovers of X and then taking the connected components of these.

Remark 7.4. Some properties that hold and can be found in [AM86].

1. $H^*(X_{\text{et}}, -) \cong H^*_{\text{et}}(X, -)$.
2. $\pi_1(X_{\text{et}}, \overline{x}_{\text{et}})$ is the enlarged fundamental group of SGA3.
3. If X is normal (or geometrically unibranch), then $\pi_n(X_{\text{et}})$ is profinite for all $n \geq 0$.

As a consequence of (2) and (3), it holds that $\pi_1(X_{\text{et}}) \cong \pi_1^{\text{et}}(X)$ for normal schemes, which means that the étale homotopy type is a true generalisation of the étale fundamental group.

8. Schmidt and Stix 2016

Henceforth, let all schemes be normal (or geometrically unibranch). For X, Y over k we again consider the natural map

$$\phi_{Y,X} : \mathrm{Isom}_k(Y, X) \to \mathrm{Isom}_{\mathrm{Ho}(\text{pro-}\mathbf{ss}_*)/k_{\text{et}}}(Y_{\text{et}}, X_{\text{et}}).$$

Hyperbolic curves are of type $K(\pi, 1)$. The first main result of the paper [SS16] is the following, which is a translation of Mochizuki's result into the language of étale homotopy.

In the statements below, let k denote a finitely generated extension of $\mathbb{Q}$.

Theorem 8.1 (3.2 of [SS16])**.** *For X, Y hyperbolic curves over k, the map $\phi_{Y,X}$ is a bijection.*

A further generalisation is the following, which does not give a bijection, but at least a split injection (with a functorial retraction).

Theorem 8.2 (4.7 of [SS16])**.** *Assume that X, Y are geometrically connected over k and that they can be embedded as locally closed subschemes into a product of hyperbolic curves. Then $\phi_{Y,X}$ is a split injection.*

This has a weakly anabelian result as a direct corollary:

Corollary 8.3 (1.3 of [SS16])**.** *For X, Y as in theorem 8.2, if $X_{\text{et}} \cong_{k_{\text{et}}} Y_{\text{et}}$, then $X \cong_k Y$.*

Remark 8.4. Schmidt shows in [Sch12] that under certain conditions, the étale homotopy type functor factors through the Morel–Voevodsky $\mathbb{A}^1$-homotopy type. In particular, this means that it would be impossible to extract more information from the étale homotopy type of a scheme than what is contained in the $\mathbb{A}^1$-homotopy type. Furthermore, anabelian schemes would have to be $\mathbb{A}^1$-local, which indeed is the case for hyperbolic curves. Hence, it could be seen as more natural to ask the more general "motivic anabelian" question: which schemes are reconstructible from the $\mathbb{A}^1$-homotopy type? ○

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